THE LOCAL LEHMER INEQUALITY FOR DRINFELD MODULES

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ABSTRACT. We give a lower bound for the local height of a non-torsion element of a Drinfeld module.

1. INTRODUCTION

In this paper we will use the following notation: p is a prime number and q is a power of p. We denote by $\mathbb{F}_q$ the finite field with q elements. We let C be a nonsingular projective curve defined over $\mathbb{F}_q$ and we fix a closed point on C, which we call ∞. Then we define A as the ring of functions on C that are regular everywhere except possibly at ∞. Also, in this paper, the elements of $\mathbb{F}_q^{\text{alg}}$ will be called constants.

We let K be a finitely generated field extension of $\mathbb{F}_q$. We fix a morphism $i : A \to K$. We define the operator τ as the power of the usual Frobenius with the property that for every $x \in K^{\text{alg}}$, $\tau(x) = x^q$. Then we let $K\{\tau\}$ be the ring of polynomials in τ with coefficients from K.

A Drinfeld module is a morphism $\phi : A \to K\{\tau\}$ for which the coefficient of τ^0 in ϕ_a is $i(a)$ for every $a \in A$. Following the definition from [3] we will call ϕ a Drinfeld module of generic characteristic if $\ker(i) = \{0\}$ and we will call ϕ a Drinfeld module of finite characteristic if $\ker(i) \neq \{0\}$. In case of a Drinfeld module of generic characteristic we will identify $i(a)$ with a for every $a \in A$.

In Section 2 we will develop the theory of heights on Drinfeld modules. We denote by $\hat{\mathrm{h}} : K^{\text{alg}} \to \mathbb{R}_{\geq 0}$ the global height associated to a Drinfeld module ϕ and for each divisor v (as defined in Section 2) we define by $\hat{\mathrm{h}}_v : K^{\text{alg}} \to \mathbb{R}_{\geq 0}$ the corresponding local height associated to ϕ.

The paper [1] proposed the following conjecture, which is the Drinfeld module analogue of the classical Lehmer inequality.

Conjecture 1.1. For the Drinfeld module $\phi : A \to K\{\tau\}$ there exists a constant $C > 0$, depending only on ϕ, such that any non-torsion point $x \in K^{\text{alg}}$ satisfies $\hat{\mathrm{h}}(x) \geq \frac{C}{[K(x):K]}$.

A partial result towards this conjecture was obtained in [2].

The following statement would imply (1.1) and we refer to it as the local case of the Lehmer inequality for Drinfeld modules.

Statement 1.2. For the Drinfeld module $\phi : A \to K\{\tau\}$ there exists a constant $C > 0$, depending only on ϕ, such that for any $x \in K^{\text{alg}}$ and any place v of $K(x)$, if $\hat{\mathrm{h}}_v(x) > 0$, then $\hat{\mathrm{h}}_v(x) \geq \frac{C}{[K(x):K]}$.

In the third section of this paper we will prove that (1.2) is false but in the case of Drinfeld modules of finite characteristic there is the following result.

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Theorem 1.3. *For $\phi : A \to K\{\tau\}$ a finite characteristic Drinfeld module, there exist two positive constants C and k depending only on ϕ such that if $x \in K^{\mathrm{alg}}$ and v is a place of $K(x)$ for which $\hat{\mathrm{h}}_v(x) > 0$, then $\hat{\mathrm{h}}_v(x) \geq \frac{C}{d^k}$ (where $d = [K(x) : K]$).*

Theorem (1.3) will follow from the following stronger result.

Theorem 1.4. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic. Let $x \in K^{\mathrm{alg}}$ and let $d = [K(x) : K]$. Let $v \in M_{K(x)}$ such that $\hat{\mathrm{h}}_v(x) > 0$. Denote by v_0 the place of K sitting below v and let $e(v|v_0)$ be the corresponding ramification index.*

There exists $C > 0$ and $k \geq 1$, both depending only on ϕ, such that $\hat{\mathrm{h}}_v(x) \geq \frac{C}{e(v|v_0)^{k-1}d}$.

Moreover if p does not divide $e(v|v_0)$, then we can give a very easy expression for the exponent k in (1.4) which will be optimal in this case as shown by example (3.6). This will be proved in theorem (3.8).

If ϕ is a Drinfeld module of generic characteristic, example (3.9) will show that $\hat{\mathrm{h}}_v(x)$ can be arbitrarily small and strictly positive regardless of $d = [K(x) : K]$. In theorem (3.10), we will give the best result towards conjecture (1.2) for Drinfeld modules of generic characteristic.

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2. Heights associated to Drinfeld modules

As stated in Section 1, we are working with a Drinfeld module $\phi : A \to K\{\tau\}$. Because K is a finitely generated field over $\mathbb{F}_q$, it is the function field of a variety V defined over $\mathbb{F}_{q^m}$ for some $m \geq 1$ and in addition we can take V to be normal and projective, embedded in $\mathbb{P}^M$ (for some $M \geq 1$). We define M_K as the set of valuations of K that are associated to irreducible divisors of V, i.e. codimension 1 subvarieties of V. Then to an element x from K, we associate its divisor

$$(x) = \sum_{\rho \in M_K} v_\rho(x) \cdot \rho$$

where by $v_\rho(x)$ we denoted the order of x at ρ.

For each $\rho \in M_K$, we denote by $\deg(\rho)$ the projective degree of ρ in $\mathbb{P}^M$, which is the intersection number of ρ with a generic $(M - N + 1)$-dimensional hyperplane in $\mathbb{P}^M$ ($N = \dim V$). The following product formula holds

$$\sum_{\rho \in M_K} \deg(\rho) \cdot v_\rho(x) = 0.$$

For simplicity of notation in the rest of this paper we will drop the index ρ from the valuation v.

Now we construct the local heights $\hat{\mathrm{h}}_v$ with respect to the Drinfeld module ϕ. Our construction follows [4] together with the observations from [5] that extend the construction to finitely generated function fields. So, for $x \in K$ and $v \in M_K$, we set $\tilde{v}(x) = \min\{0, v(x)\}$

and for a nonconstant element $a \in A$, we define

$$V_v(x) = \lim_{n\to\infty} \frac{\tilde{v}(\phi_{a^n}(x))}{\deg(\phi_{a^n})}.$$

This function satisfies the same properties as in Propositions 1-4 from [4]. We define

$$\hat{\mathrm{h}}_v(x) = -\deg(v) V_v(x)$$

where $\deg(v)$ is the degree of the divisor v as defined above.

This defines the local heights only for elements of K, but we will be interested in extending them to K^{alg}. For this, let $x \in L$, where L is a finite extension of K. We let W be the normalization of V in L and form the set M_L of valuations of L associated to W. As shown in [5], for every $v \in M_K$, there exist finitely many $w \in M_L$ extending v. When we work in such a setting, our convention will always be that the valuations are functions with range $\mathbb{Z}$. Thus $w|_K = e(w|v)v$, where $e(w|v)$ is the corresponding ramification index. We define the function V_w as V_v from above. Then we let

$$\hat{\mathrm{h}}_w(x) = -\frac{\deg(v) f(w|v)}{[L:K]} V_w(x)$$

where $f(w|v)$ is the relative residue degree between the residue field of L at w and the residue field of K at v.

Then, just as in [4], we define the global height with respect to ϕ by

$$\hat{\mathrm{h}}(x) = \sum_{w\in M_L} \hat{\mathrm{h}}_w(x).$$

The above sum is finite due to a similar argument as the one from Proposition 6 of [4].

If L' is a finite extension of L and $w'|w$ is any valuation on L' extending w, then $V_{w'}(x) = e(w'|w)V_w(x)$. Because $\sum_{w'|w} e(w'|w)f(w'|w) = [L':L]$, we get $\sum_{w'|w} \hat{\mathrm{h}}_{w'}(x) = \hat{\mathrm{h}}_w(x)$ for every $x \in L$ and every $w \in M_L$. Thus, our definition of the global height is independent of the field L containing x.

Let t be a non-constant element of A and $\phi_t = \sum_{i=r_0}^{r} a_i\tau^i$, with $a_{r_0} \neq 0$. The first results of this section do not rely on ϕ being of finite characteristic or not and so, we do not specify right now if $r_0 = 0$ or $r_0 > 0$.

In proving (1.4) we may replace K by a finite extension K'. This will only induce a constant factor $[K':K]$ in the denominator of the lower bound for the local height. Also, (1.4) is not affected if we replace ϕ by a Drinfeld module that is isomorphic to ϕ. Thus we can conjugate ϕ by an element $\gamma \in K^{\mathrm{alg}}$ such that $\phi^{(\gamma)}$, the conjugated Drinfeld module, has the property that $\phi_t^{(\gamma)}$ is monic as a polynomial in τ. Then ϕ and $\phi^{(\gamma)}$ are isomorphic over $K(\gamma)$ which is a finite extension of K (because γ satisfies the equation $\gamma^{q^r-1}a_r = 1$).

So, we will prove theorem (1.4) for $\phi^{(\gamma)}$ and because $\hat{\mathrm{h}}_{\phi,v}(x) = \hat{\mathrm{h}}_{\phi^{(\gamma)},v}(\gamma^{-1}x)$ (as proved in [4], Proposition 2) the result will follow for ϕ. For simplicity of notation we will suppose from now on that ϕ_t is monic as a polynomial in τ.

Let x be a nonzero element of K^{alg} and let $L = K(x)$. Denote by S the finite subset of M_L where the coefficients a_i, for $i \in \{r_0, \ldots, r-1\}$, have poles. Also, denote by S_0 the finite set of divisors from M_K where the coefficients a_i have poles. Thus, each divisor from S sits above an unique divisor from S_0.

For each $v \in M_L$ denote by

$$M_v = \min_{i \in \{r_0, \dots, r-1\}} \frac{v(a_i)}{q^r - q^i} \tag{1}$$

where by convention: $v(0) = +\infty$. If $r_0 = r$, definition (1) is void and in that case we define $M_v = +\infty$.

Note that $M_v < 0$ if and only if $v \in S$.

For each $v \in S$ we fix a uniformizer $\pi_v \in L$ of the place v. We define next the concept of angular component for every $y \in L \setminus \{0\}$.

Definition 2.1. Assume $v \in S$. For every nonzero $y \in L$ we define the angular component of y at v, denoted by $\mathrm{ac}_{\pi_v}(y)$, the residue at v of $y\pi_v^{-v(y)}$. (Note that the angular component is never 0.)

We can define in a similar manner as above the notion of angular component at each $v \in M_L$ but we will work with angular components at the places from S only.

The main property of the angular component is that for every $y, z \in L$, $v(y - z) > v(y) = v(z)$ if and only if $(v(y), \mathrm{ac}_{\pi_v}(y)) = (v(z), \mathrm{ac}_{\pi_v}(z))$.

Our strategy for proving (1.4) will be to prove that if $\hat{\mathrm{h}}_v(x) > 0$ then *either*

$$\hat{\mathrm{h}}_v(x) \geq \frac{C}{e(v|v_0)^{\frac{r}{r_0}-1} d}$$

where $e(v|v_0)$ is the corresponding ramification index of v over K, $d = [L : K]$ and $C > 0$ is a constant depending only on ϕ, *or*

$v \in S$ and $(v(x), \mathrm{ac}_{\pi_v}(x))$ belongs to a set of cardinality we can control.

For $v \in S$ we define

$$P_v = \left\{ \frac{v(a_i) - v(a_j)}{q^j - q^i} | r_0 \leq i < j \leq r \text{ and } a_i \neq 0 \neq a_j \right\} \cup \{0\}. \tag{2}$$

Clearly, $|P_v| \leq 1 + \binom{r - r_0 + 1}{2}$. For each $\alpha \in P_v$ we let $l \geq 1$ and let $i_0 < i_1 < \cdots < i_l$ be all the indices i for which $a_i \neq 0$ and moreover, for $j, k \in \{0, \dots, l\}$ with $j \neq k$, we have

$$\frac{v(a_{i_j}) - v(a_{i_k})}{q^{i_k} - q^{i_j}} = \alpha. \tag{3}$$

We define $R_v(\alpha)$ as the set containing $\{1\}$ and all the nonzero solutions of the equation

$$\sum_{j=0}^{l} \mathrm{ac}_{\pi_v}(a_{i_j}) X^{q^{i_j}} = 0. \tag{4}$$

Clearly, for every $\alpha \in P_v$, $|R_v(\alpha)| \leq q^r$.

Note that if $\alpha = 0$, there might be no indices i_j and i_k as in (3). In that case, the construction of $R_v(0)$ from (4) is void and so, we define $R_v(0) = \{1\}$. The motivation for the special case $0 \in P_v$ and $1 \in R_v(0)$ is explained in the proof of lemma (2.7).

Lemma 2.2. *Assume $v \in S$. If $v(\phi_t(x)) > \min_{i \in \{r_0, \dots, r\}} v(a_i x^{q^i})$ then $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$.*

Proof. If $v(x) > \min_{i\in\{r_0,\ldots,r\}} v(a_i x^{q^i})$ it means that there exists $l \geq 1$ and $i_0 < \cdots < i_l$ such that

$$v(a_{i_0}x^{q^{i_0}}) = \cdots = v(a_{i_l}x^{q^{i_l}}) \tag{5}$$

and also

$$\sum_{j=0}^{l} \mathrm{ac}_{\pi_v}(a_{i_j})\, \mathrm{ac}_{\pi_v}(x)^{q^{i_j}} = 0. \tag{6}$$

Equations (5) and (6) yield $v(x) \in P_v$ and $\mathrm{ac}_{\pi_v}(x) \in R_v(v(x))$ respectively, according to (2) and (4). □

Lemma 2.3. *Let $v \in M_L$ and let $v_0 \in M_K$ be the unique valuation of K sitting below v. If $v(x) < \min\{0, M_v\}$, then $\hat{\mathrm{h}}_v(x) = \frac{-\deg(v_0)f(v|v_0)}{[L:K]}v(x)$.*

Proof. For every $i \in \{r_0,\ldots,r-1\}$, $v(a_i x^{q^i}) = v(a_i) + q^i v(x) > q^r v(x)$ because $v(x) < M_v = \min_{i\in\{r_0,\ldots,r-1\}} \frac{v(a_i)}{q^r - q^i}$. This shows that $v(\phi_t(x)) = q^r v(x) < v(x) < \min\{0, M_v\}$. By induction, $v(\phi_{t^n}(x)) = q^{rn}v(x)$ for all $n \geq 1$. So, $V_v(x) = v(x)$ and

$$\hat{\mathrm{h}}_v(x) = \frac{-\deg(v_0)f(v|v_0)}{[L:K]}v(x).$$

□

An immediate corollary to (2.3) is the following result.

Lemma 2.4. *Assume $v \notin S$. If $v(x) < 0$ then $\hat{\mathrm{h}}_v(x) = \frac{-\deg(v_0)f(v|v_0)}{[L:K]}v(x)$, while if $v(x) \geq 0$ then $\hat{\mathrm{h}}_v(x) = 0$.*

Proof. First, it is clear that if $v(x) \geq 0$ then for all $n \geq 1$, $v(\phi_{t^n}(x)) \geq 0$ because all the coefficients of ϕ_t and thus of ϕ_{t^n} have non-negative valuation at v. Thus, $V_v(x) = 0$ and so

$$\hat{\mathrm{h}}_v(x) = 0.$$

Now, if $v(x) < 0$, then $v(x) < M_v$ because $M_v \geq 0$ ($v \notin S$). So, applying the result of (2.3) we conclude the proof of this lemma. □

We will get a better insight into the local heights behaviour with the following lemma.

Lemma 2.5. *Assume $v \in S$ and $v(x) \leq 0$. If $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ then $v(\phi_t(x)) < M_v$, unless $q = 2$, $r = 1$ and $v(x) = 0$.*

Proof. Lemma (2.2) implies that there exists $i_0 \in \{r_0,\ldots,r\}$ such that for all $i \in \{r_0,\ldots,r\}$ we have $v(a_i x^{q^i}) \geq v(a_{i_0}x^{q^{i_0}}) = v(\phi_t(x))$.

Suppose (2.5) is not true and so, there exists $j_0 < r$ such that

$$\frac{v(a_{j_0})}{q^r - q^{j_0}} \leq v(\phi_t(x)) = v(a_{i_0}) + q^{i_0}v(x).$$

This means that

$$v(a_{j_0}) \leq (q^r - q^{j_0})v(a_{i_0}) + (q^{r+i_0} - q^{i_0+j_0})v(x). \tag{7}$$

On the other hand, by our assumption about i_0, we know that $v(a_{j_0}x^{q^{j_0}}) \geq v(a_{i_0}x^{q^{i_0}})$ which means that

$$v(a_{j_0}) \geq v(a_{i_0}) + (q^{i_0} - q^{j_0})v(x). \tag{8}$$

Putting together inequalities (7) and (8), we get

$$v(a_{i_0}) + (q^{i_0} - q^{j_0})v(x) \leq (q^r - q^{j_0})v(a_{i_0}) + (q^{r+i_0} - q^{i_0+j_0})v(x).$$

Thus

$$v(x)(q^{r+i_0} - q^{i_0+j_0} - q^{i_0} + q^{j_0}) \geq -v(a_{i_0})(q^r - q^{j_0} - 1). \tag{9}$$

But $q^{r+i_0} - q^{i_0+j_0} - q^{i_0} + q^{j_0} = q^{r+i_0}(1 - q^{j_0-r} - q^{-r} + q^{j_0-r-i_0})$ and because $j_0 < r$ and $q^{j_0-r-i_0} > 0$, we obtain

$$1 - q^{j_0-r} - q^{-r} + q^{j_0-r-i_0} > 1 - q^{-1} - q^{-r} \geq 1 - 2q^{-1} \geq 0. \tag{10}$$

Also, $q^r - q^{j_0} - 1 \geq q^r - q^{r-1} - 1 = q^{r-1}(q-1) - 1 \geq 0$ with equality if and only if $q = 2$, $r = 1$ and $j_0 = 0$. We will analyze this case separately. So, as long as we are not in this special case, we do have

$$q^r - q^{j_0} - 1 > 0. \tag{11}$$

Now we have two possibilities (remember that $v(x) \leq 0$):

(i) $v(x) < 0$

In this case, (9), (10) and (11) tell us that $-v(a_{i_0}) < 0$. Thus, $v(a_{i_0}) > 0$. But we know from our hypothesis on i_0 that $v(a_{i_0}x^{q^{i_0}}) \leq v(x^{q^r})$ which is in contradiction with the combination of the following facts: $v(x) < 0$, $i_0 \leq r$ and $v(a_{i_0}) > 0$.

(ii) $v(x) = 0$

Then another use of (9), (10) and (11) gives us $-v(a_{i_0}) \leq 0$; thus $v(a_{i_0}) \geq 0$. This would mean that $v(a_{i_0}x^{q^{i_0}}) \geq 0$ and this contradicts our choice for i_0 because we know from the fact that $v \in S$, that there exists $i \in \{r_0, \ldots, r\}$ such that $v(a_i) < 0$. So, then we would have

$$v(a_ix^{q^i}) = v(a_i) < 0 \leq v(a_{i_0}x^{q^{i_0}}).$$

Thus, in either case (i) or (ii) we get a contradiction that proves the lemma except in the special case that we excluded above: $q = 2$, $r = 1$ and $j_0 = 0$. If we have $q = 2$ and $r = 1$ then

$$\phi_t(x) = a_0x + x^2.$$

Note that if $a_0 \in \mathbb{F}_p^{\mathrm{alg}}$, S is empty and so, the result of our lemma is vacuously true. Thus, we suppose from now on that in this case: $q = 2$ and $r = 1$, $a_0 \notin \mathbb{F}_p^{\mathrm{alg}}$ and so, S consists of the irreducible divisors of the pole of a_0.

If $v(x) \leq 0$, then either $v(x) < M_v = v(a_0)$, in which case again $v(\phi_t(x)) < M_v$ (as shown in the proof of lemma (2.3)), or $v(x) \geq M_v$ and so, $i_0 = 0$ (because in this case $v(a_0x) \leq v(x^2)$). In the latter case,

$$v(\phi_t(x)) = v(a_0x) = v(a_0) + v(x) < v(a_0) = M_v$$

unless $v(x) = 0$. So, we see that indeed, only $v(x) = 0$, $q = 2$ and $r = 1$ can make $v(\phi_t(x)) \geq M_v$ in the hypothesis of (2.5). $\square$

Lemma 2.6. *Assume $v \in S$. Excluding the case $q = 2$, $r = 1$ and $v(x) = 0$, we have that if $v(x) \leq 0$ then either $\hat{\mathrm{h}}_v(x) > \frac{-\deg(v_0)f(v|v_0)M_v}{q^r[L:K]}$ or $(v(x), ac_{\pi_v}(x)) \in P_v \times R_v(v(x))$.*

Proof. If $v(x) \leq 0$ then

$$\textit{either}\text{: (i) } v(\phi_t(x)) < M_v \ ,$$

in which case by (2.3) we have that $\hat{\mathrm{h}}_v(\phi_t(x)) = \frac{-\deg(v_0)f(v|v_0)}{[L:K]} v(\phi_t(x))$. So, case (i) yields

$$\hat{\mathrm{h}}_v(x) = \frac{-\deg(v_0)f(v|v_0)}{[L:K]} \cdot \frac{v(\phi_t(x))}{\deg \phi_t} > \frac{-\deg(v_0)f(v|v_0)}{[L:K]} \cdot \frac{M_v}{q^r} \tag{12}$$

$$\textit{or}\text{: (ii) } v(\phi_t(x)) \geq M_v \ ,$$

in which case, lemma (2.5) yields

$$v(\phi_t(x)) > v(a_{i_0}x^{q^{i_0}}) = \min_{i \in \{r_0 \dots, r\}} v(a_i x^{q^i}). \tag{13}$$

Using (13) and lemma (2.2) we conclude that case (ii) yields $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$. ☐

Now we analyze the excluded case from lemma (2.6).

Lemma 2.7. *Assume $v \in S$. If $v(x) \leq 0$ then either $(v(x), ac_{\pi_v}(x)) \in P_v \times R_v(v(x))$ or $\hat{\mathrm{h}}_v(x) \geq \frac{-\deg(v_0)f(v|v_0)M_v}{q^r[L:K]}$.*

Proof. Using the result of (2.6) we have left to analyze the case: $q = 2$, $r = 1$ and $v(x) = 0$. As shown in the proof of (2.5), in this case $\phi_t(x) = a_0x + x^2$ and

$$v(\phi_t(x)) = v(a_0) = M_v < 0.$$

Then, either $v(\phi_{t^2}(x)) = v(\phi_t(x)^2) = 2M_v < M_v$ or $v(\phi_{t^2}(x)) > v(a_0\phi_t(x)) = v(\phi_t(x)^2)$. If the former case holds, then by (2.3),

$$\hat{\mathrm{h}}_v(\phi_{t^2}(x)) = \frac{-\deg(v_0)f(v|v_0)}{[L:K]} \cdot 2M_v \Rightarrow \hat{\mathrm{h}}_v(x) = \frac{-\deg(v_0)f(v|v_0)}{[L:K]} \frac{2M_v}{4}.$$

If the latter case holds, i.e. $v(\phi_t(\phi_t(x))) > v(a_0\phi_t(x)) = v(\phi_t(x)^2)$, it means that $\mathrm{ac}_{\pi_v}(\phi_t(x))$ satisfies the equation

$$\mathrm{ac}_{\pi_v}(a_0)X + X^2 = 0.$$

Because the angular component is never 0, it must be that $\mathrm{ac}_{\pi_v}(\phi_t(x)) = \mathrm{ac}_{\pi_v}(a_0)$ (remember that we are working now in characteristic 2). But, because $v(a_0x) < v(x^2)$ we can relate the angular component of x and the angular component of $\phi_t(x)$ and so,

$$ac_{\pi_v}(a_0) = \mathrm{ac}_{\pi_v}(\phi_t(x)) = \mathrm{ac}_{\pi_v}(a_0x) = ac_{\pi_v}(a_0)\,\mathrm{ac}_{\pi_v}(x).$$

This means $\mathrm{ac}_{\pi_v}(x) = 1$ and so, the excluded case amounts to a dichotomy similar to the one from (2.6): either $(v(x), \mathrm{ac}_{\pi_v}(x)) = (0, 1)$ or $\hat{\mathrm{h}}_v(x) = \frac{\deg(v_0)f(v|v_0)}{[L:K]} \frac{-M_v}{2}$. The definitions of P_v and $R_v(\alpha)$ from (2) and (4) respectively, yield that $(0, 1) \in P_v \times R_v(0)$. ☐

Finally, we note that in (2.7) we have

$$-\deg(v_0)f(v|v_0)M_v \geq -M_v > \frac{e(v|v_0)}{q^r}.$$

We have obtained the following dichotomy.

Lemma 2.8. *Assume $v \in S$. If $v(x) \le 0$ then either $\hat{\mathrm{h}}_v(x) \ge \frac{e(v|v_0)}{q^{2r}d}$ or $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$ with $|P_v| \le 1 + \binom{r-r_0+1}{2}$ and for each $\alpha \in P_v$, $|R_v(\alpha)| \le q^r$.*

Lemma 2.9. *There are no x and x' verifying the following properties*
(a) $v(x) \neq v(x')$;
(b) $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ and $(v(x'), \mathrm{ac}_{\pi_v}(x')) \notin P_v \times R_v(v(x'))$;
(c) $v(\phi_t(x)) = v(\phi_t(x'))$.

Proof. Suppose (2.9) is not true and so, there exist x, x' satisfying (a),(b) and (c). Property (b) and lemma (2.2) yield that there exists $i_1 \in \{r_0, \dots, r\}$ such that

$$v(\phi_t(x)) = v(a_{i_1}) + q^{i_1} v(x) \tag{14}$$

and for every $i \in \{r_0, \dots, r\}$,

$$v(a_{i_1}) + q^{i_1} v(x) \le v(a_i) + q^i v(x). \tag{15}$$

We have the similar equations for x' and for some $i_2 \in \{r_0, \dots, r\}$,

$$v(\phi_t(x')) = v(a_{i_2}) + q^{i_2} v(x') \tag{16}$$

where for every $i \in \{r_0, \dots, r\}$,

$$v(a_{i_2}) + q^{i_2} v(x') \le v(a_i) + q^i v(x'). \tag{17}$$

Also, we supposed that we have

$$v(\phi_t(x)) = v(\phi_t(x')). \tag{18}$$

We assume that $i_1 \neq i_2$, because $i_1 = i_2$ would imply from (14), (16) and (18) that $v(x) = v(x')$. So, without loss of generality we may assume that $i_1 < i_2$. We use (15) for $i = i_2$ and so, we get $v(a_{i_1}) + q^{i_1} v(x) \le v(a_{i_2}) + q^{i_2} v(x)$ which implies

$$v(x) \ge \frac{v(a_{i_1}) - v(a_{i_2})}{q^{i_2} - q^{i_1}}. \tag{19}$$

Now, using (17) with $i = i_1$, we get $v(a_{i_2}) + q^{i_2} v(x') \le v(a_{i_1}) + q^{i_1} v(x')$ which implies

$$v(x') \le \frac{v(a_{i_1}) - v(a_{i_2})}{q^{i_2} - q^{i_1}}. \tag{20}$$

But because of (18) together with (14) and (16), we have $v(a_{i_1}) + q^{i_1} v(x) = v(a_{i_2}) + q^{i_2} v(x')$ which implies that

$$v(x') = \frac{q^{i_1} v(x) + v(a_{i_1}) - v(a_{i_2})}{q^{i_2}}. \tag{21}$$

Using (20) and (21), we get

$$\frac{q^{i_1} v(x) + v(a_{i_1}) - v(a_{i_2})}{q^{i_2}} \le \frac{v(a_{i_1}) - v(a_{i_2})}{q^{i_2} - q^{i_1}}.$$

So, $q^{i_1}(q^{i_2} - q^{i_1}) v(x) \le q^{i_1}(v(a_{i_1}) - v(a_{i_2}))$, which implies that

$$v(x) \le \frac{v(a_{i_1}) - v(a_{i_2})}{q^{i_2} - q^{i_1}}$$

which combined with (19) shows that

$$v(x) = \frac{v(a_{i_1}) - v(a_{i_2})}{q^{i_2} - q^{i_1}}.$$

Then, using (21) we get

$$v(x') = \frac{q^{i_1}\frac{v(a_{i_1})-v(a_{i_2})}{q^{i_2}-q^{i_1}} + v(a_{i_1}) - v(a_{i_2})}{q^{i_2}} = \frac{v(a_{i_1}) - v(a_{i_2})}{q^{i_2} - q^{i_1}} = v(x)$$

which shows that indeed $v(\phi_t(x))$ can be obtained from an unique value for $v(x)$. $\square$

Lemma 2.10. *Assume $v \in S$. If $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ then for each of the values $(\alpha_1, \gamma_1) = (v(\phi_t(x)), \mathrm{ac}_{\pi_v}(\phi_t(x)))$ there are at most q^r possible values γ for $\mathrm{ac}_{\pi_v}(x)$.*

Proof. Indeed, we saw in lemma (2.9) that $v(x)$ is uniquely determined given $\alpha_1 = v(\phi_t(x))$ under the hypothesis of (2.10). We also have

$$\mathrm{ac}_{\pi_v}(\phi_t(x)) = \sum_j \mathrm{ac}_{\pi_v}(a_{i_j})\,\mathrm{ac}_{\pi_v}(x)^{q^{i_j}} \tag{22}$$

where i_j runs through a prescribed subset of $\{r_0, \ldots, r\}$ corresponding to those i such that $v(a_i) + q^i v(x) = v(\phi_t(x))$. This subset of indices i_j, depends only on $\alpha_1 = v(x)$. So, there are at most q^r possible values for $\mathrm{ac}_{\pi_v}(x)$ to solve (22) given $\gamma_1 = \mathrm{ac}_{\pi_v}(\phi_t(x))$. $\square$

From now on in this section we will suppose that

$\boxed{r_0 \geq 1, \text{ i.e. } \phi \text{ has finite characteristic and } t \in A \text{ has the property that } \phi_t \text{ is inseparable.}}$

Lemma 2.11. *For $v \in S$ denote by $N_v = \max\left\{\frac{-v(a_i)}{q^i-1} \mid 1 \leq i \leq r\right\}$ (remember our convention on $v(0) = +\infty$). If $v(x) \geq N_v$, then $\hat{\mathrm{h}}_v(x) = 0$.*

Proof. Indeed, if $v(x) \geq N_v$ then $v(\phi_t(x)) \geq \min_{1\leq i\leq r}\{q^i v(x) + v(a_i)\} \geq v(x) \geq N_v$. By induction, we get that $v(\phi_{t^n}(x)) \geq N_v$ for all $n \geq 1$, which yields that $V_v(x) = 0$ and so,

$$\hat{h}_v(x) = 0.$$

$\square$

Thus, if $v \in S$ and $\hat{\mathrm{h}}_v(x) > 0$ it must be that $v(x) < N_v$.

Lemma 2.12. *Assume $v \in S$. If $v(x) < N_v$ and if $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ then $v(\phi_t(x)) < v(x)$.*

Proof. Indeed, by the hypothesis and by lemma (2.2), there exists $i_0 \in \{r_0, \ldots, r\}$ such that for all $i \in \{r_0, \ldots, r\}$,

$$v(a_{i_0}) + q^{i_0}v(x) = v(\phi_t(x)) \leq v(a_i) + q^i v(x). \tag{23}$$

If $v(\phi_t(x)) \geq v(x)$ then, using (23), we get that

$$v(x) \leq v(a_i) + q^i v(x)$$

which implies that $v(x) \geq -\frac{v(a_i)}{q^i-1}$ for every i. Thus

$$v(x) \geq N_v,$$

contradicting the hypothesis of our lemma. So, we must have $v(\phi_t(x)) < v(x)$. In particular, we also get that $v(a_{i_0}) + q^{i_0} v(x) < v(x)$, i.e.

$$v(x) < \frac{-v(a_{i_0})}{q^{i_0} - 1}. \tag{24}$$

□

Our goal is establishing a dichotomy similar to the one from lemma (2.8) under the following hypothesis

$\boxed{v \in S,\ \hat{\mathrm{h}}_v(x) > 0 \text{ and } 0 < v(x) < N_v.}$

In lemma (2.12) we saw that if $v(x) < N_v$ then either $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$ or $v(\phi_t(x)) < v(x)$. In the latter case, if $v(\phi_t(x)) > 0$ we apply then the same reasoning to $\phi_t(x)$ and derive that either $(v(\phi_t(x)), \mathrm{ac}_{\pi_v}(\phi_t(x))) \in P_v \times R_v(v(\phi_t(x)))$ or $v(\phi_{t^2}) < v(\phi_t(x))$. We repeat this analysis and after a finite number of steps, say n, we must have that either $v(\phi_{t^n}(x)) \leq 0$ or $(v(\phi_{t^n}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^n}(x))) \in P_v \times R_v(v(\phi_{t^n}(x)))$. But we analyzed in (2.8) what happens to the cases in which, for an element y of positive local height at v, $v(y) \leq 0$. We obtained that either

$$\hat{h}_v(y) \geq \frac{e(v|v_0)}{q^{2r} d} \tag{25}$$

or

$$(v(y), \mathrm{ac}_{\pi_v}(y)) \in P_v \times R_v(v(y)) \tag{26}$$

and $|P_v| \leq 1 + \binom{r-r_0+1}{2} \leq 1 + \frac{r^2-r}{2} = \frac{r^2-r+2}{2}$ because $r_0 \geq 1$.

We will use repeatedly equations (25) and (26) for $y = \phi_{t^n}(x)$. So, if (25) holds for $y = \phi_{t^n}(x)$ then

$$\hat{h}_v(x) \geq \frac{e(v|v_0)}{q^{rn} q^{2r} d}. \tag{27}$$

We will see next what happens if (26) holds. We can go back through the steps that we made in order to get to (26) and see that actually $v(x)$ and $\mathrm{ac}_{\pi_v}(x)$ belong to prescribed sets of cardinality independent of n.

Lemma 2.13. *Assume $v \in S$ and suppose that $v(x) < N_v$. If $(v(\phi_{t^k}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^k}(x))) \notin P_v \times R_v(v(\phi_{t^k}(x)))$ for $0 \leq k \leq n-1$, then for each value $(\alpha_n, \gamma_n) = (v(\phi_{t^n}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^n}(x)))$, $v(x)$ is uniquely determined and $\mathrm{ac}_{\pi_v}(x)$ belongs to a set of cardinality at most $q^{r \cdot \binom{r}{2}}$.*

Proof. The fact that $v(x)$ is uniquely determined follows after n successive applications of lemma (2.9) to $\phi_{t^{n-1}}(x), \ldots, \phi_t(x), x$.

Because $(v(\phi_{t^k}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^k}(x))) \notin P_v \times R_v(v(\phi_{t^k}(x)))$ for $k < n$, it means that we are solving an equation of the form

$$\sum_j \mathrm{ac}_{\pi_v}(a_{i_j})\, \mathrm{ac}_{\pi_v}(\phi_{t^k}(x))^{q^{i_j}} = \mathrm{ac}_{\pi_v}(\phi_{t^{k+1}}(x)) \tag{28}$$

in order to express $\mathrm{ac}_{\pi_v}(\phi_{t^k}(x))$ in terms of $\mathrm{ac}_{\pi_v}(\phi_{t^{k+1}}(x))$ for each $k < n$. The equations (28) are uniquely determined by the sets of indices $i_j \in \{r_0, \ldots, r\}$ which in turn are uniquely

determined by $v(\phi_{t^k}(x))$, i.e. for each k and each corresponding index i_j

$$v(a_{i_j}\phi_{t^k}(x)^{q^{i_j}}) = \min_{i\in\{r_0,\dots,r\}} v(a_i\phi_{t^k}(x)^{q^i}). \tag{29}$$

Using the result of (2.12) and the hypothesis of our lemma, we see that

$$v(x) > v(\phi_t(x)) > v(\phi_{t^2}(x)) > \cdots > v(\phi_{t^n}(x)) \tag{30}$$

and so the equations from (28) appear in a prescribed order. Now, in most of the cases, these equations will consist of only one term on their left-hand side; i.e. they will look like

$$\mathrm{ac}_{\pi_v}(a_{i_0})\,\mathrm{ac}_{\pi_v}(\phi_{t^k}(x))^{q^{i_0}} = \mathrm{ac}_{\pi_v}(\phi_{t^{k+1}}(x)). \tag{31}$$

Equation (31) has an unique solution. The other equations of type (28) but not of type (31) are associated to some of the values of $v(\phi_{t^k}(x)) \in P_v$. Indeed, according to the definition of P_v from (2), only for those values we can have for $i \neq i'$

$$v(a_i) + q^i v(x) = v(a_{i'}) + q^{i'} v(x) \tag{32}$$

and so, both indices i and i' can appear in (28).

So, the number of equations of type (28) but not of type (31) is at most $\binom{r}{2}$ (remember that we are working under the assumption that ϕ_t is inseparable, i.e. $r_0 \geq 1$). Moreover these equations will appear in a prescribed order, each not more than once, because of (30). These observations determine the construction of the finite set that will contain all the possible values for $\mathrm{ac}_{\pi_v}(x)$, given $\gamma_n = \mathrm{ac}_{\pi_v}(\phi_{t^n}(x)))$. An equation of type (28) can have at most q^r solutions; thus $\mathrm{ac}_{\pi_v}(x)$ lives in a set of cardinality at most $q^{r\cdot\binom{r}{2}}$. □

Because of the result of (2.13), we know that we can construct in an unique way $v(x)$ given $v(\phi_{t^n}(x))$ and the fact that for every $j < n$, $\phi_{t^j}(x)$ does not satisfy (26). So, for each n there are at most $|P_v|$ values for $v(x)$ such that

$$(v(\phi_{t^n}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^n}(x))) \in P_v \times R_v(v(\phi_{t^n}(x))) \tag{33}$$

where n is minimal with this property. We denote by $P_v(n)$ this set of values for $v(x)$. By convention: $P_v(0) = P_v$.

Also, lemma (2.13) yields that for each fixed $(v(\phi_{t^n}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^n}(x))) \in P_v \times R_v(v(\phi_{t^n}(x)))$ there are at most

$$q^{r\cdot\binom{r}{2}} = q^{\frac{r^3-r^2}{2}} \tag{34}$$

possibilities for $\mathrm{ac}_{\pi_v}(x)$. For $\alpha = v(x) \in P_v(n)$ we define by $R_v(\alpha)$ the set of all possible values for $\mathrm{ac}_{\pi_v}(x)$ such that (33) holds. Let $v(\phi_{t^n}(x)) = \alpha_n \in P_v$ and using the definition of $R_v(\alpha_n)$ for $\alpha_n \in P_v$ from (4), we get

$$|R_v((v(\phi_{t^n}(x)))| \leq q^r. \tag{35}$$

Inequality (35) and the result of lemma (2.13) gives the estimate:

$$|R_v(\alpha)| \leq |R_v(v(\phi_{t^n}(x)))| \cdot q^{\frac{r^3-r^2}{2}} \leq q^r \cdot q^{\frac{r^3-r^2}{2}} = q^{\frac{r^3-r^2+2r}{2}} \tag{36}$$

for every $\alpha \in P_v(n)$ and for every $n \geq 0$.

Now, we estimate the magnitude of n, i.e. the number of steps that we need to make starting with $0 < v(x) < N_v$ such that in the end $\phi_{t^n}(x)$ satisfies either (25) or (26).

Lemma 2.14. *Assume $v \in S$ and $\hat{\mathrm{h}}_v(x) > 0$. Then there exists a set P of cardinality bounded in terms of r and $e(v|v_0)$ such that either $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P \times R_v(v(x))$ or $\hat{\mathrm{h}}_v(x) > \frac{c_1}{e(v|v_0)^{\frac{r}{r_0}-1} d}$ with $c_1 > 0$ depending only on ϕ.*

Proof. If (26) does not hold for x then we know that there exists $i_0 \geq r_0$ such that $v(\phi_t(x)) = q^{i_0} v(x) + v(a_{i_0})$.

Now, if $\phi_t(x)$ also does not satisfy (26) then for some i_1

$$v(\phi_{t^2}(x)) = q^{i_1} v(x) + v(a_{i_1}) \leq q^i v(\phi_t(x)) + v(a_i)$$

for all $i \in \{r_0, \ldots, r\}$. So, in particular

$$v(\phi_{t^2}(x)) \leq q^{i_0} v(\phi_t(x)) + v(a_{i_0}) \tag{37}$$

and in general

$$v(\phi_{t^{k+1}}(x)) \leq q^{i_0} v(\phi_{t^k}(x)) + v(a_{i_0}) \tag{38}$$

if $(v(\phi_{t^k}(x)), \mathrm{ac}_{\pi_v}(\phi_{t^k}(x))) \notin P_v \times R_v(v(\phi_{t^k}(x)))$. Let us define the following sequence $(y_j)_{j \geq 0}$ by

$$y_0 = v(x) \text{ and for all } j \geq 1\text{: } y_j = q^{i_0} y_{j-1} + v(a_{i_0}).$$

If $\phi_{t^i}(x)$ does not satisfy (26) for $i \in \{0, \ldots, n-1\}$ then by (38),

$$y_n \geq v(\phi_{t^n}(x)). \tag{39}$$

The sequence $(y_j)_{j \geq 0}$ can be easily computed and we see that

$$y_j = q^{i_0 j} \left(v(x) + \frac{v(a_{i_0})}{q^{i_0} - 1} \right) - \frac{v(a_{i_0})}{q^{i_0} - 1}. \tag{40}$$

But $v(x) < -\frac{v(a_{i_0})}{q^{i_0}-1}$, as a consequence of $v(x) < N_v$ and the proof of lemma (2.12) (see equation (24)). Thus,

$$v(x) + \frac{v(a_{i_0})}{q^{i_0} - 1} \leq -\frac{1}{q^{i_0} - 1} \tag{41}$$

because $v(x), v(a_{i_0}) \in \mathbb{Z}$. Using inequality (41) in the formula (40) we get

$$y_j \leq \frac{1}{q^{i_0} - 1} (-q^{i_0 j} - v(a_{i_0})). \tag{42}$$

For v_0 the valuation of K that sits under the valuation v of L, we define

$$c_{v_0} = \max \left\{ -v_0(a_i) | r_0 \leq i \leq r \right\}. \tag{43}$$

So, $c_{v_0} \geq 1$ because we know that at least one of the a_i has a pole at v, thus at v_0 (we are working under the assumption that $v \in S$). Clearly, c_{v_0} depends only on ϕ and on K; thus, for simplicity we denote c_{v_0} by c in the next calculations. Because of the definition of c, we have

$$-v(a_{i_0}) \leq e(v|v_0) c \tag{44}$$

where $e(v|v_0)$ is as always the ramification index of v over v_0. Now, if we pick m minimal such that

$$q^{r_0 m} \geq c e(v|v_0) \tag{45}$$

then we see that m depends only on ϕ and $e(v|v_0)$. Using that $i_0 \geq r_0$ we get that

$$q^{i_0 m} \geq ce(v|v_0).$$

So, using inequalities (42), (44) and (45) we obtain $y_m \leq 0$. Because of (39) we derive that

$$v(\phi_{t^m}(x)) \leq 0$$

which according to the dichotomy from lemma (2.8) yields that $\phi_{t^m}(x)$ satisfies either (25) or (26). Thus, we need at most m steps to get from x to some $\phi_{t^n}(x)$ for which one of the two equations (25) or (26) is valid. This means that either

$$\hat{h}_v(x) \geq \frac{e(v|v_0)}{q^{rm}q^{2r}d} \text{ (which holds if (25) is valid after } n \leq m \text{ steps),} \tag{46}$$

or

$$\phi_{t^n}(x) \text{ satisfies (26) for } n \leq m. \tag{47}$$

This last equation implies that $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v(n) \times R_v(v(x))$ for some $n \leq m$.

We analyze now the inequality from equation (46). By the minimality of m satisfying (45), we have

$$q^{rm} = (q^{r_0(m-1)})^{\frac{r}{r_0}} q^r < (ce(v|v_0))^{\frac{r}{r_0}} q^r. \tag{48}$$

So, if (46) holds, we have the following inequality

$$\hat{h}_v(x) > \frac{e(v|v_0)}{c^{\frac{r}{r_0}} q^{3r} e(v|v_0)^{\frac{r}{r_0}} d}. \tag{49}$$

We denote by $P = \bigcup_{i=0}^m P_v(i)$. We proved that for $i \geq 1$, $|P_v(i)| \leq |P_v(0)|$ (and $P_v = P_v(0)$ has cardinality depending only on r; this was mainly the content of (2.13)). To simplify the notations in the future we introduce new constants c_i, that will always depend only on ϕ. For example, (49) says that

$$\hat{h}_v(x) > \frac{c_1}{e(v|v_0)^{\frac{r}{r_0}-1} d} \text{ or } (v(x), \mathrm{ac}_{\pi_v}(x)) \in P \times R_v(v(x)) \tag{50}$$

and $|R_v(v(x))| \leq q^{\frac{r^3-r^2+2r}{2}}$ (see equation (36)), while $|P| \leq \frac{r^2-r+2}{2}(m+1)$ with m satisfying (48). ☐

3. The local Lehmer inequality

We continue with the notation from Section 2. The field L is finitely generated and $v \in M_L$. First we will prove the following general lemma on valuations.

Lemma 3.1. *Let I be a finite set of integers. Let N be an integer greater or equal than all the elements of I. For each $\alpha \in I$, let $R(\alpha)$ be a nonempty finite set of nonzero elements of the residue field at v. Let W be an $\mathbb{F}_q$-vector subspace of L with the property that for all $0 \neq w \in W$, $(v(w), \mathrm{ac}_{\pi_v}(w)) \in I \times R(v(w))$ whenever $v(w) \leq N$.*

Let f be the smallest integer greater or equal than $\max_{\alpha \in I} \log_q |R(\alpha)|$. Then the codimension of $\{w \in W \mid v(w) > N\}$ is bounded by $|I|f$.

Proof. To prove (3.1) it suffices to show the following statement.

Claim 3.2. We cannot find a subspace $W' \subset W$ of dimension $1 + |I|f$ such that for all $0 \neq w \in W'$, $(v(w), \mathrm{ac}_{\pi_v}(w)) \in I \times R(v(w))$.

To prove claim (3.2) we will use induction on $z = |I|$.

If $z = 1$, then $I = \{\alpha\}$. Assume that there are $(1 + f)$ $\mathbb{F}_q$-linearly independent elements

$$w_1, w_2, \dots, w_{f+1}$$

such that for all $0 \neq w \in \mathrm{Span}(\{w_1, \dots, w_{f+1}\})$, $v(w) = \alpha$ and $\mathrm{ac}_{\pi_v}(w) \in R(\alpha)$. By our choice for f, we have more nonzero $\mathbb{F}_q$-linear combinations of

$$\mathrm{ac}_{\pi_v}(w_1), \dots, \mathrm{ac}_{\pi_v}(w_{f+1})$$

than elements of $R(\alpha)$. Thus, there exists an $\mathbb{F}_q$-linear combination

$$\gamma = d_1 \,\mathrm{ac}_{\pi_v}(w_1) + \dots + d_{f+1} \,\mathrm{ac}_{\pi_v}(w_{f+1})$$

where not all $d_1, \dots, d_{f+1}$ are 0 and, either $0 \neq \gamma \notin R(\alpha)$ or $\gamma = 0$. So, if we let

$$w = d_1 w_1 + \dots + d_{f+1} w_{f+1}$$

then, either $\mathrm{ac}_{\pi_v}(w) \notin R(\alpha)$ or $v(w) > \alpha$. Thus, claim (3.2) holds for $z = 1$.

Now we prove the inductive step. We assume that our claim (3.2) holds for z, with $z \geq 1$, and we prove it for $(z + 1)$.

We assume $I = \{\alpha_1, \alpha_2, \dots, \alpha_{z+1}\}$ and we suppose there exist $(1 + (z + 1)f)$ $\mathbb{F}_q$-linearly independent elements of W, $w_1, \dots, w_{1+(z+1)f}$ such that for all nonzero

$$w \in \mathrm{Span}(\{w_1, \dots, w_{1+(z+1)f}\})$$

we have that $(v(w), \mathrm{ac}_{\pi_v}(w)) \in I \times R(v(w))$.

By the induction hypothesis we know that there are no $(1 + zf)$ indices

$$i_j \in \{1, \dots, 1 + (z + 1)f\}$$

such that for each such index, $v(w_{i_j}) \geq \alpha_2$. Thus, without loss of generality we may assume that there exists $1 \leq g \leq 1 + zf$ such that $v(w_i) = \alpha_1$ for all $i \in \{1, \dots, f + g\}$, while $v(\alpha_i) \geq \alpha_2$ if $i > f + g$. If $g = 1 + zf$, then there are no indices i such that $v(w_i) \geq \alpha_2$.

There are two cases:

(i) $\dim_{\mathbb{F}_q} \mathrm{Span}(\{\mathrm{ac}_{\pi_v}(w_1), \dots, \mathrm{ac}_{\pi_v}(w_{f+g})\}) > f$.

In this case, the definition of f from (3.1) yields the existence of a nonzero

$$w \in \mathrm{Span}(\{w_1, \dots, w_{f+g}\})$$

such that $v(w) = \alpha_1$ and $\mathrm{ac}_{\pi_v}(w) \notin R(\alpha_1)$. This proves the claim (3.2) if (i) holds.

(ii) $\dim_{\mathbb{F}_q} \mathrm{Span}(\{\mathrm{ac}_{\pi_v}(w_1), \dots, \mathrm{ac}_{\pi_v}(w_{f+g})\}) \leq f$.

In this case, without loss of generality we may assume that

$$\mathrm{Span}(\{ac_{\pi_v}(w_1), \dots, \mathrm{ac}_{\pi_v}(w_{f+g})\}) = \mathrm{Span}(\{ac_{\pi_v}(w_1), \dots, \mathrm{ac}_{\pi_v}(w_f)\}).$$

Then for every $k \in \{1, \dots, g\}$ we can find $l_k \in \mathrm{Span}(\{w_1, \dots, w_f\})$ such that $v(l_k) = \alpha_1$ and $\mathrm{ac}_{\pi_v}(l_k) = \mathrm{ac}_{\pi_v}(w_{f+k})$. Thus

$$v(w_{f+k} - l_k) > \alpha_1 \tag{51}$$

Because $0 \neq w_{f+k} - l_k \in \mathrm{Span}(\{w_1, \dots, w_{1+(z+1)f}\})$ for every $k \in \{1, \dots, g\}$, it means that

$$v(w_{f+k} - l_k) \in I. \tag{52}$$

Equations (51) and (52) yield that $v(w_{f+k} - l_k) \geq \alpha_2$. This shows that for all $0 \neq w \in \mathrm{Span}(\{w_{f+1} - l_1, \dots, w_{f+g} - l_g, w_{f+g+1}, \dots, w_{1+(z+1)f}\})$,

$$v(w) \in \{\alpha_2, \dots, \alpha_{z+1}\} \text{ and } \mathrm{ac}_{\pi_v}(w) \in R(v(w)).$$

Because all of the $(1+zf)$ elements $w_{f+1}-l_1,\dots,w_{f+g}-l_g,w_{f+g+1},\dots,w_{1+(z+1)f}$ are $\mathbb{F}_q$-linearly independent, we get a contradiction with the inductive hypothesis of claim (3.2). This concludes the proof of lemma (3.1). $\square$

We are ready to prove theorem (1.4).

Proof of Theorem 1.4. First we observe that if $v\notin S$ then by lemma (2.4) we automatically get the lower bound $\hat{\mathrm{h}}_v(x)\geq\frac{1}{d}$ because it must be that $v(x)<0$, otherwise we would have $\hat{\mathrm{h}}_v(x)=0$. So, from now on we suppose that the valuation v is from S.

We denote by $z=|P|$. Let f be the smallest integer such that

$$f\geq\max_{\alpha\in P}\log_q|R_v(\alpha)|.$$

So $f\leq\frac{r^3-r^2+2r}{2}$, as shown by the proof of lemma (2.14). We also have the following inequality

$$zf\leq\frac{r^2-r+2}{2}(m+1)\cdot\frac{r^3-r^2+2r}{2}=\frac{r^5-2r^4+5r^3-4r^2+4r}{4}(m+1). \tag{53}$$

Let $W=\mathrm{Span}(\{x,\phi_t(x),\dots,\phi_{t^{zf}}(x)\})$. Because $\hat{\mathrm{h}}_v(x)>0$ we know that $x\notin\phi_{\mathrm{tor}}$ and so, $\dim_{\mathbb{F}_q}W=1+zf$. We also get from $\hat{\mathrm{h}}_v(x)>0$ that for all $0\neq w\in W$, $\hat{\mathrm{h}}_v(w)>0$. Then by lemma (2.11), we get that for all $0\neq w\in W$, $v(w)\leq N_v-1$.

We apply lemma (3.1) to W with $I=P$, $R=R_v$, $N=N_v-1$ and conclude that there exists $0\neq b\in\mathbb{F}_q[t]$, of degree at most zf in t such that

$$(v(\phi_b(x)),\mathrm{ac}_{\pi_v}(\phi_b(x)))\notin P\times R_v(v(\phi_b(x))). \tag{54}$$

We know that $\hat{h}_v(x)>0$ and so $\hat{h}_v(\phi_b(x))>0$. Equations (54) and (50) yield

$$\hat{h}_v(\phi_b(x))>\frac{c_1}{e(v|v_0)^{\frac{r}{r_0}-1}d}.$$

Thus

$$\hat{h}_v(x)>\frac{c_1}{q^{r\deg(b)}e(v|v_0)^{\frac{r}{r_0}-1}d}.$$

But, using inequality (53), we obtain

$$q^{r\deg(b)}\leq q^{rzf}\leq q^{\frac{r^5-2r^4+5r^3-4r^2+4r}{4}r(m+1)}=q^{\frac{r^6-2r^5+5r^4-4r^3+4r^2}{4}}(q^{rm})^{\frac{r^5-2r^4+5r^3-4r^2+4r}{4}}.$$

We use (48) and we get

$$q^{r\deg(b)}<q^{\frac{r^6-2r^5+5r^4-4r^3+4r^2}{4}}(ce(v|v_0))^{\frac{r}{r_0}\cdot\frac{r^5-2r^4+5r^3-4r^2+4r}{4}}q^{\frac{r^6-2r^5+5r^4-4r^3+4r^2}{4}}.$$

Thus there exists a constant $C>0$ depending only on c_1, c, q and r such that

$$\hat{h}_v(x)>\frac{C}{e(v|v_0)^{\frac{r}{r_0}(\frac{r^5-2r^4+5r^3-4r^2+4r}{4}+1)-1}d}. \tag{55}$$

Because c_1 and c depend only on ϕ we get the conclusion of (1.4). $\square$

Using that $e(v|v_0)\leq d$, we get the conclusion of theorem (1.3)

$$\hat{h}(x)\geq\frac{C}{d^k},$$

with $k\leq\frac{r^6-2r^5+5r^4-4r^3+4r^2+4r}{4r_0}$.

Remark 3.3. From the above proof we see that the constant C depends only on q, r and the numbers $v(a_i)$ for $r_0 \leq i \leq r-1$, in the hypothesis that ϕ_t is monic as a polynommial in τ. As we said before, for the general case, when ϕ_t is not neccessarily monic, the constant C from (1.4) will be multiplied by the inverse of the degree of the extension of K that we have to allow in order to construct a conjugated Drinfeld module $\phi^{(\gamma)}$ for which $\phi_t^{(\gamma)}$ is monic. The degree of this extension is at most (q^r-1) because $\gamma^{q^r-1}a_r = 1$.

Remark 3.4. It is interesting to note that (55) shows that the original statement of (1.2) holds, i.e. $k=1$, in the case that $e(v|v_0)=1$, which is the case when x belongs to an unramified extension above v_0. Also, as observed in the beginning of the proof of (1.4), if v and so, equivalently v_0 is not a pole for any of the a_i then we automatically get exponent $k=1$ in theorem (1.4), as proved in lemma (2.4).

So, we see that in the course of proving (1.4) we got an even stronger result that allows us to conclude that conjecture (1.2) and so, implicitly conjecture (1.1) holds in the maximal unramified extension above the finitely many irreducible divisors from S_0.

Remark 3.5. Also, it is interesting to note that the above proof shows that for every divisor v associated to L (as in Section 2), there exists a number n depending only on r and $e(v|v_0)$ so that there exists $b \in \mathbb{F}_q[t]$ of degree at most n in t for which either $v(\phi_b(x)) < M_v$ (in which case $\hat{\mathrm{h}}_v(x) > 0$), or $v(\phi_b(x)) \geq N_v$ (in which case $\hat{\mathrm{h}}_v(x) = 0$).

Example 3.6. The result of theorem (1.4) is optimal in the sense that we cannot hope to get the conjectured Lehmer inequality for the local height, i.e. $\frac{C}{d}$. We can only get, in the general case for the local height, an inequality with some exponent $k>1$, i.e. $\frac{C}{d^k}$.

For example, take $A = \mathbb{F}_q[t]$ and define

$$\phi_t = \tau^r - t^{1-q}\tau.$$

Let $K = \mathbb{F}_q(t)$. Let $d = q^m - 1$, for some $m \geq r$. Then let $x = t\alpha$ where α is a root of

$$\alpha^d - \alpha - \frac{1}{t} = 0.$$

Then $L = K(x)$ is totally ramified above t of degree d. Let v be the unique valuation of L for which $v(t) = d$. We compute

$$P_v = \left\{ \frac{-d(q-1)}{q^r - q} \right\}$$

$$M_v = -\frac{d(q-1)}{q^r-q}$$

$$N_v = d$$

$$v(x) = d - 1 = q^m - 2.$$

We compute easily $v(\phi_{t^i}(x)) = d - q^i$ for every $i \in \{0, \ldots, m\}$.

Furthermore, $v(\phi_{t^m}(x)) = d - q^m = -1 \neq \frac{-d(q-1)}{q^r-q}$, because $\frac{-d(q-1)}{q^r-q} \notin \mathbb{Z}$ $(q \not| d(q-1))$. Thus $v(\phi_{t^m}(x))$ is negative and not in P_v and so, (2.5) yields

$$v(\phi_{t^{m+1}}(x)) < M_v.$$

Actually, because $m \geq r$, an easy computation shows that

$$v\left(\frac{\phi_{t^m}(x)^q}{t^{q-1}}\right) = -q - d(q-1) = -q^{m+1} + q^m - 1 < -q^r = v((\phi_{t^m}(x))^{q^r}).$$

This shows that $v(\phi_{t^{m+1}}(x)) = -q^{m+1} + q^m - 1 < M_v < 0$ and so, by (2.3)

$$\hat{\mathrm{h}}_v(x) = \frac{\hat{\mathrm{h}}_v(\phi_{t^{m+1}}(x))}{q^{r(m+1)}} = \frac{q^{m+1} - q^m + 1}{q^{r(m+1)}d} < \frac{q^{m+1}}{q^{m+r}q^{(r-1)m}d} < \frac{q^{1-r}}{d^r},$$

because $d = q^m - 1 < q^m$.

This computation shows that for Drinfeld modules of type

$$\phi_t = \tau^r - t^{1-q}\tau$$

the exponent k from (1.4) should be at least r. The exact same computation will give us that in the case of a Drinfeld module of the form

$$\phi_t = \tau^r - t^{1-q^{r_0}}\tau^{r_0}$$

for some $1 \leq r_0 < r$ and x of valuation $(q^{r_0m} - 2)$ at a place v that is totally ramified above the place of t with ramification index $q^{r_0m} - 1$, the exponent k in theorem (1.4) should be at least $\frac{r}{r_0}$. In theorem (3.8) we will prove that for non-wildly ramified extensions above places from S_0, we indeed get exponent $k = \frac{r}{r_0}$. But before doing this, we observe that the present example is just a counter-example to statement (1.2), not to conjecture (1.1). In other words, the global Lehmer inequality holds for our example even if the local one fails.

Indeed, because x was chosen to have positive valuation at the only place from S, it means that there exists another place, call it v' which is not in S, for which $v'(x) < 0$. But then by lemma (2.4), we get that $\hat{\mathrm{h}}_{v'}(x) \geq \frac{1}{d}$, which means that also $\hat{\mathrm{h}}(x) \geq \frac{1}{d}$. Thus we obtain a lower bound for the global height as conjectured in (1.1).

Now, in order to get to the result of (3.8) we prove a lemma.

Lemma 3.7. *With the notation from the proof of theorem* (1.4)*, let* $L = \mathrm{lcm}_{i\in\{1,\dots,r-r_0\}}\{q^i - 1\}$. *If* p *does not divide* $e(v|v_0)$ *then* $e(v|v_0)$ *divides* $L\alpha$ *for every* $\alpha \in P$.

Proof. Indeed, from its definition (2), P_v contained $\{0\}$ and numbers of the form

$$\frac{v(a_i) - v(a_j)}{q^j - q^i} = \frac{v(a_i) - v(a_j)}{q^i(q^{j-i} - 1)},$$

for $j > i$. Clearly, every number of this form, times L is divisible by $e(v|v_0)$, because we supposed that $p \not| e(v|v_0)$. The set $P_v(1)$ contains numbers of the form

$$\frac{\alpha - v(a_i)}{q^i} \tag{56}$$

where $\alpha \in P_v = P_v(0)$ and $a_i \neq 0$. Using again that p does not divide $e(v|v_0)$ we get that $e(v|v_0) \mid L\alpha_1$ for all $\alpha_1 \in P_v(1)$. Repeating the process from (56) we obtain all the elements of $P_v(n)$ for every $n \geq 1$ and by induction on n, we conclude that $e(v|v_0) \mid L\alpha_n$ for all $\alpha_n \in P_v(n)$. Because $P = \bigcup_{n=0}^{m} P_v(n)$ we get the result of this lemma. $\square$

Theorem 3.8. *Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of finite characteristic. Let $t \in A$ such that $\phi_t = \sum_{i=1}^{r} a_i\tau^i$ is inseparable. Let r_0 the index of the first nonzero coefficient of ϕ_t. Let $x \in K^{\mathrm{alg}}$ and let $v \in M_{K(x)}$ such that $h_v(x) > 0$. Let v_0 be the valuation on K that sits below v.*

If p does not divide $e(v|v_0)$, there exists a constant $C > 0$ depending only on ϕ such that $\hat{h}_v(x) \geq \frac{C}{e(v|v_0)^{\frac{r}{r_0}-1}[K(x):K]}$.

Proof. Just as we observed in Section 2 and in remark (3.3), it suffices to prove (3.8) under the hypothesis that ϕ_t is monic in τ.

Let now $d = [K(x) : K]$. We observe again that from (2.4) it follows that if $v \notin S$ then $\hat{\mathrm{h}}_v(x) \geq \frac{1}{d} \geq \frac{1}{e(v|v_0)^{\frac{r}{r_0}-1}d}$. So, from now on we consider the case $v \in S$.

Then, using the result of (3.7) in (24) we see that

$$v(x) + \frac{v(a_{i_0})}{q^{i_0} - 1} \leq -\frac{\frac{e(v|v_0)}{L}}{q^{i_0} - 1} \tag{57}$$

if $v(x) \in P$. Then also (42) changes into

$$x_m \leq \frac{1}{q^{i_0} - 1}(-q^{i_0 m}\frac{e(v|w)}{L} - v(a_{i_0})). \tag{58}$$

So, then we choose m' minimal such that

$$q^{r_0 m'} \geq cL \tag{59}$$

where $c = c_{v_0}$ is the same as in (43). Thus m' depends only on ϕ. We redo the computations from (46) to (50), this time with m' in place of m and because of (58) and (59), we get that

$$\hat{h}_v(x) > \frac{c_1}{e(v|v_0)^{\frac{r}{r_0}-1}d} \text{ or } (v(x), \mathrm{ac}_{\pi_v}(x)) \in P' \times R_v(v(x)) \tag{60}$$

where $P' = \bigcup_{i=0}^{m'} P_v(i)$. At this moment we can redo the argument from the proof of (1.4) using P' instead of P, only that now $z' = |P'|$ is independent of x or d. We conclude once again that there exists b, a polynomial in t of degree at most $z'f$ such that

$$\hat{h}_v(\phi_b(x)) > \frac{c_1}{e(v|v_0)^{\frac{r}{r_0}-1}d}.$$

But because both f and z' depend only on ϕ, we conclude that indeed,

$$\hat{h}(x) \geq \frac{C}{e(v|v_0)^{\frac{r}{r_0}-1}d}$$

with $C > 0$ depending only on ϕ. $\square$

Example 3.9. We discuss now the conjecture (1.2) for Drinfeld modules of generic characteristic. So, consider the Carlitz defined on $\mathbb{F}_p[t]$ by $\phi_t = t\tau^0 + \tau$, where $\tau(x) = x^p$ for all x. Take $K = \mathbb{F}_p(t)$. Let L be a finite extension of K which is totally ramified above ∞ and so, let the ramification index equals $d = [L : K]$. Also, let v be the unique valuation of L sitting above ∞.

Let $x \in L$ of valuation nd at v for some $n \geq 1$. An easy computation shows that for all $m \in \{1, \ldots, n\}$, $v(\phi_{t^m}(x)) = dn - dm$. So, in particular $v(\phi_{t^n}(x)) = 0$ and so,

$$v(\phi_{t^{n+1}}(x)) = -d < M_v = \frac{-d}{p-1} .$$

This shows, after using lemma (2.3), that $\hat{\mathrm{h}}_v(\phi_{t^{n+1}}(x)) = \frac{d}{d} = 1$. This in turn implies that

$$\hat{\mathrm{h}}_v(x) = \frac{1}{p^{n+1}}.$$

But we can take n arbitrarily large, which shows that there is no way to obtain a similar result like theorem (1.4) for generic characteristic Drinfeld modules.

The next theorem shows that the example (3.9) is in some sense the only way theorem (1.4) fails for Drinfeld modules of generic characteristic.

Theorem 3.10. *Let ϕ be a Drinfeld module of generic characteristic and so, with the usual notation, let $\phi_t = t\tau^0 + \sum_{i=1}^{r} a_i\tau^i$, for a non-constant $t \in A$. Let $x \in K^{\mathrm{alg}}$ and let v be an irreducible divisor from $M_{K(x)}$ that does not sit over the place ∞ from* $\mathrm{Frac}(A)$. *Let $v_0 \in M_K$ sit below v. There exist two positive constants C and k depending only on ϕ, such that if $\hat{\mathrm{h}}_v(x) > 0$ then $\hat{\mathrm{h}}_v(x) \geq \frac{C}{e(v|v_0)^{k-1}[K(x):K]}$.*

Proof. Again, as we mentioned in Section 2 and in remark (3.3), it suffices to prove this theorem under the hypothesis that ϕ_t is monic in τ. Also, if $v \notin S$ theorem (3.10) holds as shown by lemma (2.4).

The analysis of local heights done in Section 2 applies to both finite and generic characteristic until lemma (2.11). So, we still get the conclusion of lemma (2.8). Thus, if $v(x) \leq 0$ then either $\hat{\mathrm{h}}_v(x) \geq \frac{e(v|v_0)}{q^{2r}[K(x):K]}$ or $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$, with $|P_v|$ and $|R_v(v(x))|$ depending only on q and r.

We know from our hypothesis that $v(t) \geq 0$ and so,

$$v(tx) \geq v(x). \tag{61}$$

Now, if $v(x) \geq N_v$, then $v(a_i x^{q^i}) \geq v(x)$, for all $i \geq 1$ (by the definition of N_v) and using also equation (61), we get

$$v(\phi_t(x)) \geq v(x) \geq N_v.$$

Iterating this computation we get that $v(\phi_{t^n}(x)) \geq N_v$, for all $n \geq 1$ and so $\hat{\mathrm{h}}_v(x) = 0$, contradicting the hypothesis of our theorem. This argument is the equivalent of lemma (2.11) for Drinfeld modules of generic characteristic under the hypothesis $v(t) \geq 0$.

Thus it must be that $v(x) < N_v$. Then, lemma (2.12) holds identically. This yields that either $(v(x), \mathrm{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$ or $v(\phi_t(x)) < v(x)$.

From this point on, the proof continues just as for theorem (1.4). We form just as before the sets $P_v(n)$ and their union will be again denoted by P. We conclude once again as in (49) that *either*

$$\hat{\mathrm{h}}_v(x) \geq \frac{1}{q^{3r} c^{\frac{r}{r_0}} e(v|v_0)^{\frac{r}{r_0}-1}[K(x):K]}$$

with the same $c > 0$ depending only on q, r and ϕ as in the proof of (1.4), *or*

$$(v(x), \mathrm{ac}_{\pi_v}(x)) \in P \times R_v(v(x))$$

where $|P|$ is of the order of $\log e(v|v_0)$. We observe that when we use equations (38), (40), (41), (42) the index i_0 is still at least 1. This is the case because if $v(x) < N_v$ and $(v(x), \mathrm{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ then there exists $i_0 \geq 1$ such that $v(\phi_t(x)) = v(a_{i_0}) + q^{i_0} v(x)$. Also, $v(x) < N_v$ means that there exists at least one index $i \in \{1, \ldots, r\}$ such that $v(tx) \geq v(x) > v(a_i x^{q^i})$.

Thus, the first index other than 0 of a non-zero coefficient a_i will play the role of r_0 as in the proof of (1.4). Finally, lemma (3.1) finishes the proof of theorem (3.10). $\square$

So, we get in the same way as in the proof of (1.4), the conclusion for theorem (3.10). The difference made by v not sitting above ∞ is that for $v(x) \geq 0$, $v(\phi_t(x))$ can decrease only if $v(x) < N_v$, i.e. only if there exists $i \geq 1$ such that $v(a_i x^{q^i}) < v(x)$. If v sits over ∞, then $v(tx) < v(x)$ and so, $v(\phi_t(x))$ might decrease just because of the $t\tau^0$ term from ϕ_t. Thus, in that case, as example (3.9) showed, we can start with x having arbitrarily large valuation and we are able to decrease it by applying ϕ_t to it repeatedly, making the valuation of $\phi_{t^n}(x)$ be less than M_v, which would mean that $\hat{\mathrm{h}}_v(x) > 0$. But in doing this we will need a number n of steps (of applying ϕ_t) that we will not be able to control; so $\hat{\mathrm{h}}_v(x)$ will be arbitrarily small.

It is easy to see that remarks (3.4) and (3.5) are valid also for theorem (3.10) in the hypothesis that v does not sit over the place ∞ of $\mathrm{Frac}(A)$. Also, just as we were able to derive theorem (3.8) from the proof of (1.4), we can do the same thing in theorem (3.10) and find a specific value of the constant k that will work in the case that v is not wildly ramified above $v_0 \in M_K$. The result is the following theorem whose proof goes along the same lines as the proof of (3.8).

Theorem 3.11. *Let ϕ be a Drinfeld module of generic characteristic and let $\phi_t = t\tau^0 + \sum_{i=r_0}^{r} a_i \tau^i$, with $a_{r_0} \neq 0$ (of course, $r_0 \geq 1$). There exists a constant $C > 0$, depending only on ϕ such that for every $x \in K^{\mathrm{alg}}$ and every $v \in M_{K(x)}$ that is neither wildly ramified above K nor sitting above the place ∞ of $\mathrm{Frac}(A)$, if $\hat{\mathrm{h}}_v(x) > 0$ then $\hat{\mathrm{h}}_v(x) \geq \frac{C}{e(v|v_0)^{\frac{r}{r_0}-1}[K(x):K]}$.*

We can also construct an example simliar to (3.6) which shows that constant $k = \frac{r}{r_0}$ is optimal in the above theorem. Indeed, if we take a Drinfeld module ϕ defined on $\mathbb{F}_q[t]$ by

$$\phi_t = t\tau^0 + \frac{1}{t^{q^{r_0}-1}}\tau^{r_0} + \tau^r$$

and x as in example (3.6) then a similar computation will show that we cannot hope for an exponent k smaller than $\frac{r}{r_0}$.

The constants C from theorem (3.8), (3.10) and (3.11) and the constant k from (3.10) have the same corresponding dependency on q, r and ϕ as explained in the proof of theorem (1.4).

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